Ultracold dipolar gases - a challenge for experiments and theory

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We present a review of recent results concerning the physics of ultracold trapped dipolar gases. In particular, we discuss the Bose-Einstein condensation for dipolar Bose gases and the BCS transition for dipolar Fermi gases. In both cases we stress the dominant role of the trap geometry in determining the properties of the system. We present also results concerning bosonic dipolar gases in optical lattices and the possibility of obtaining variety of different quantum phases in such case. Finally, we analyze various possible routes towards achieving ultracold dipolar gases.

I. INTRODUCTION

In the last twenty years at least three Nobel Prizes were awarded for studies of the phenomena of condensation, superfluidity and superconductivity *, whereas several others were also related to these subjects [1]. The three Nobel Prizes in atomic, molecular and optical (AMO) physics within the same period mark the path from trapping and cooling of ions, through laser cooling of atoms, towards Bose-Einstein condensation (BEC) of trapped atomic gases [2]. The observation of BEC [3] has bursted a new interdisciplinary area of modern AMO and condensed matter physics: the physics of ultracold *weakly interacting* trapped quantum gases [4]. So far, most of the experimental results in this area can be very accurately modeled by mean-field methods and its extensions, based on the Gross-Pitaevskii (GP) equation and Bogoliubov-de Gennes equations for bosonic gases [5], and on the BCS theory for fermionic ones [6].

This new research area presents several novel aspects, in particular the finite character and inhomogeneity of the considered systems, and perhaps more important, the unprecedented possibility of control and manipulation of the system properties, which allows to study situations that were not yet encountered in condensed matter or low temperature physics. The number of parameters which can be successfully controlled is large, e.g. temperature, number of atoms, trap potentials, interatomic interactions, etc. In addition, the internal level structure of the atoms can also be employed, e.g. to manipulate BECs using Raman-Bragg techniques [7], or to analyze multicomponent condensates [8]

It is also worth mentioning, that the experimental techniques have progressed recently to a stage at which mean field methods cease to provide a proper physical picture. In this sense, recent experiments at JILA [9] in which the scattering length can be modified at will by using Feshbach resonances, allow to realize systems with very large scattering length, in which the mean-field picture is no more applicable. Similarly, the realization of a Bose-condensed metastable Helium gas [10], with a potential to study higher order correlation functions of the system, demands an analysis beyond the mean field theory. The recent demonstration of the Mott insulator to superfluid phase transition with atoms in an optical lattice [11], predicted in Ref. [12], belongs to the same category, but at the same time opens a new research area of AMO physics: the physics of *strongly correlated* quantum gases. The system in question allows for an easy and accurate control and manipulation, and thus provides a new and particularly promising test ground for theories of quantum phase transitions [13], which have traditionally dealt with condensed-matter rather than atomic systems.

In recent years considerable interest has been devoted to another aspect of the internal structure of the particles forming an ultracold gas, namely their dipole moment. If such dipole moment is sufficiently large, the resulting dipole-dipole forces may influence, or even completely change the properties of BEC in bosonic gases, the conditions for BCS transition in fermionic gases, or the phase diagram for quantum phase transitions for ultracold dipolar gases in optical lattices. In this paper we present a review of our recent results on ultracold dipolar gases. We consider here only the case of polarized dipolar gases when all dipoles are oriented in the same direction, and discuss the effects of the dipole-dipole interactions and their interplay with short range (Van der Waals) interactions. We do not consider several other important aspects of dipole-dipole interactions, e.g. the role of the dipole-dipole interactions in the spontaneous polarization of spinor condensates in optical lattices [14], or the self-

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bound structures in the field of a traveling wave [15]. As dipolar interactions can be quite strong (relative to the typical low-energy collisions characterized by the s-wave scattering length), the dipolar particles are considered as promising candidates for the implementation of fast and robust quantum-computing schemes [16,17].

Our review is organized as follows. Section II is devoted to bosonic dipolar gases, the prospects for their condensation, the ground-state properties of the BEC, and its elementary excitations in a trap. In sections III, an analogous analysis is applied to ultracold dipolar Fermi gases. In particular, the possibility of the BCS transition is investigated. Section IV briefly considers the issue of quantum phase transitions in a dipolar Bose gas placed in an optical lattice. In section V we discuss various possible physical systems which could lead to an ultracold dipolar gas. Finally, we conclude in section VI.

II. ULTRACOLD DIPOLAR BOSE GASES

In this section, we consider a system of N bosonic particles possessing a dipole moment, and confined in a harmonic trap of cylindrical symmetry. We constraint our analysis to the case in which all dipoles are assumed to be oriented along the symmetry axis of the trap. Additionally, we assume that the particles interact via dipole-dipole forces, and that these forces either play a dominant role, or at least compete with the short-range forces.

Why are *dipolar* gases interesting? The answer is simple: because they are *dipolar*. The dipole-dipole interaction potential between two dipolar particles is given by

$$V_d(\mathbf{R}) = (d^2/R^3)(1 - 3\cos^2\theta),$$

where d characterizes the dipole moment, $\mathbf{R}$ is the vector between the dipoles ($R = |\mathbf{R}|$ being its length), and θ the angle between $\mathbf{R}$ and the dipole orientation (Fig. 1). The potential $V_d(\mathbf{R})$ has two important properties: it is *anisotropic*, and is of *long-range* character. As we discuss below, these properties have important consequences.

At low temperatures one expects the dipolar Bose gases to condense. One expects also that the condensate properties will dramatically depend on the geometry of the trap. In cigar-shape traps along the dipole direction the interactions will be mainly attractive, and the condensate will be unstable, similar to the case of a gas with attractive short-range interactions (negative s-wave scattering length) [18]. Conversely, in pancake traps the interactions will be mainly repulsive, and the gas might become stable. Therefore, the dipolar gases offer the unprecedented possibility of modifying the atom-atom interactions by turning an "easy knob", namely the trap geometry, which is relatively easy to control and modify experimentally.

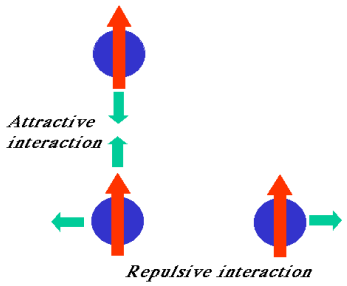

FIG. 1. Anisotropy of dipole-dipole interactions.

Several groups have recently studied the physics of ultracold dipolar bosonic gases. L. You and his collaborators [19–21] have considered the question of the validity of the mean-field approach in this case. This is a non trivial question, since contrary to the case of short-range potentials, where the s-wave scattering always becomes dominant at very low temperatures, the scattering amplitude for dipole-dipole interactions has non vanishing contributions of all partial waves at low energies. Moreover, the interplay play between the dipole interactions and the short range forces might lead to shape resonances in the atom-atom scattering. Close to such resonances the effective scattering length diverges, and the mean field approach cannot be used. Away from shape resonances, however, the Gross-Pitaevskii (GP) equation, including the non-local dipole-dipole interactions, provides a good description of the condensate. Both L. You and his coworkers [19], Góral *et al.* [22] and Mackie *et al.* [23] have investigated the properties of the ground state of these systems, and the interplay between dipolar forces and the short-range interactions. We have concentrated in our studies [24] on the case of dominant dipole-dipole interactions. This case is discussed in the next subsection.

A. Ground state properties of dipolar Bose gases

Similarly to [19,22], we describe the dynamics of the condensate wave function $\psi(\mathbf{r},t)$ by using the time-dependent GPE:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left\{-\frac{\hbar^2}{2m}\nabla^2 + \frac{m}{2}(\omega_\rho^2\rho^2 + \omega_z^2 z^2) + \right.$$
$$\left. + g|\psi(\mathbf{r},t)|^2 + d^2\int d\mathbf{r}'\frac{1-3\cos^2\theta}{|\mathbf{r}-\mathbf{r}'|^3}|\psi(\mathbf{r}',t)|^2\right\}\psi(\mathbf{r},t). \quad (1)$$

Here $\psi(\mathbf{r},t)$ is normalized to the total number of condensate particles N. The third term in the rhs is the mean-field corresponding to the short-range forces, whereas the last term is the mean field of the dipole-dipole interaction. In the following, we omit the term $g|\psi(\mathbf{r},t)|^2\psi(\mathbf{r},t)$,

assuming that the interparticle interaction is dominated by the dipole-dipole forces ($d^2 \gg |g| = 4\pi\hbar^2|a|/m$, where a is the s-wave scattering length, and m is the particle mass), and that the system is away from shape resonances of $V_d(\mathbf{R})$. The ground-state properties are governed by the stationary GPE, in which the lhs of Eq. (1) is replaced by $\mu\psi(\mathbf{r})$, where the chemical potential μ corresponds to the minimal energy solution.

There are two important parameters that can be easily controlled in experiments: the collective dipole strength Nd^2, and the trap aspect ratio $l = (\omega_\rho/\omega_z)^{1/2} = a_z/a_\rho$, where the characteristic harmonic oscillator length $a_i = \sqrt{\hbar/m\omega_i}$. The first of these parameters can be rewritten into a dimensionless form $\sigma = (Nd^2/a_{\max}^3)/\hbar\omega_{\max}$, where $a_{\max} = (\hbar/2M\omega_{\min})^{1/2}$, with ω_{min} the minimum trap frequency. Therefore, σ represents the ratio of the dipole-dipole interaction energy at the characteristic harmonic oscillator distance to the characteristic harmonic oscillator energy, ω_{min}. Another quantity, which critically characterized the state of the system is the mean dipole-dipole interaction energy per particle given by the expression $V = (1/N)\int V_d(\mathbf{r}-\mathbf{r'})\psi_0^2(\mathbf{r})\psi_0^2(\mathbf{r'})d\mathbf{r}d\mathbf{r'}$.

The results of Ref. [24] can be summarized as follows. For cigar shaped traps with $l \geq 1$ the mean-field dipole-dipole interaction is always attractive, and the gas becomes always unstable if the number of particles N exceeds a critical value N_c, which depends only on the trap aspect ratio l. The quantity $|V|$ increases with N and the shape of the cloud changes. In spherical traps the cloud becomes more elongated in the axial direction and near $N = N_c$ the shape of the cloud is close to Gaussian with the cloud aspect ratio $L = L_z/L_\rho \simeq 2.1$. In cigar-shaped traps ($l \gg 1$) especially interesting is the regime where $\hbar\omega_z \ll |V| \ll \hbar\omega_\rho$. In this case the radial shape of the cloud remains the same Gaussian as in a non-interacting gas, but the axial behavior of the condensate will be governed by the dipole-dipole interaction which acquires a quasi 1-dimensional (1D) character. Thus, one has a (quasi) 1D gas with attractive interparticle interactions, i.e. a stable (bright) soliton-like condensate, where attractive forces are compensated by the kinetic energy [25]. With increasing N, L_z decreases. Near $N = N_c$, where $|V|$ is close to $\hbar\omega_\rho$, the axial shape of the cloud also becomes Gaussian and the aspect ratio takes the value $L \approx 3.0$.

The situation is quite different for pancake traps ($l \leq 1$); in particular there exists a critical aspect ratio $l_* \simeq 0.41$, which splits the pancake traps into two different categories: soft pancake traps ($l_* \leq l < 1$) and hard pancake traps ($l < l_*$). For soft pancake traps the dipole-dipole interaction energy is positive for a small number of particles and increases with N. The quantity V reaches a maximum, and a further increase in N reduces V and makes the cloud less pancake. For a critical number of particles $N = N_c$ the BEC becomes unstable.

We have found generally that the dipolar condensate is unstable and collapses when $N > N_c$ for $V < 0$ with $|V| > \hbar\omega_\rho$. However, for hard pancake traps with $l < l_*$ the condensate is stable at any N, because V always remains positive. For small N the shape of the cloud is Gaussian in all directions. With increasing N, the quantity V increases and the cloud first becomes Thomas-Fermi in the radial direction and then, for a very large N, also axially. The ratio of the axial to radial size of the cloud, $L = L_z/L_\rho$, continuously decreases with increasing number of particles and reaches a limiting value at $N \to \infty$. In this respect, for a very large N we have a pancake Thomas-Fermi condensate.

It is worth stressing that many of the above results, calculated from a direct numerical simulation of the time-independent GPE, were also analytically obtained with the help of a variational method already used in the context of short-range interacting BEC (c.f. [26,27]).

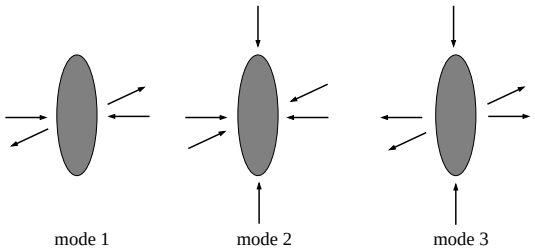

FIG. 2. Graphical representation of oscillations modes of the condensate.

B. Excitations in dipolar Bose gases

In this section we analyze the elementary excitations in a trapped dipolar gas. One possible method of analysis of these excitations could be to solve the corresponding Bogoliubov-de Gennes equations [5], which become non-local due to the presence of dipole-dipole interactions . Another approach, suggested in the context of short-range interacting condensates by Ruprecht *et al.* [28], is to use the time-dependent GPE, and numerically study the spectrum of the small perturbations around the ground-state solution. For the lowest lying excitation modes, such those schematically sketched in Fig. 2, it is also possible to obtain analytic results using the time dependent variational method of Ref. [29], which provides very accurate results when σ is not to close to its critical value σ_c, which is defined as the dipole strength at which the BEC becomes unstable. L. You and his collaborators [20,21] have recently employed this method to analyze various properties of the excitations in dipolar Bose gases. We have recently [30] used the numerical method of Ref. [28] and the variational approach of Ref. [29], in order to answer two fundamental questions concerning elementary excitations, namely how is the qualitative nature of the instability, and how the effects of

the dipole-dipole interactions can be observable from the excitation spectrum.

The question of the nature of instability is twofold. First, the determination of which of the modes becomes unstable when the number of atoms reaches the critical value. Second, the analysis of the behavior of the corresponding mode frequency close to the criticality. This analysis has been first performed for the case of a Bose gas with attractive short-range interactions. Bergeman [31] observed from his numerical results that, as the ratio of the nonlinear interaction energy to the trap frequency, γ, approaches the critical value, the frequency of the "breathing" mode 2 tends to zero and merges with the frequency of the Goldstone mode corresponding to the overall phase of the condensate. The parameter γ is defined as

$$\gamma = \sqrt{\frac{2}{\pi}} \frac{a}{\bar{\sigma}_0} N, \tag{2}$$

where $\bar{\sigma}_0$ is the geometric mean width of the ground state in an ellipsoidal harmonic trap. In the interesting region of $a < 0$, the critical value of γ is $\gamma_c = -0.54$. Above criticality, the "breathing" becomes unstable and attains complex frequency. Singh and Rokhsar analyzed this instability [32] using self-similar solutions to describe the modes, or equivalently the variational approach of the previous subsection. They have shown that close to criticality the frequency of the mode 2 vanishes as $|\gamma - \gamma_c|^{1/4}$.

In the case of dipolar gases with dominant dipole interactions the situation is completely different and much more complex. Only for aspect ratios far above the criticality, $l \gg l^*$ ($l > 1.29$) the situation reminds that of a gas with attractive short-range interactions. The mode corresponding to the lowest frequency is the "breathing" mode 2. This mode becomes unstable when the parameter $\sigma \to \sigma_c$. The scaling behavior of its frequency ω_2 can be analyzed employing the variational approach of the previous section, and the analytic (although approximate) expressions for ω_2 from Ref. [20]. We find that ω_2 goes to zero as $(\sigma_c - \sigma)^\beta$, with $\beta \simeq 1/4$.

For intermediate values of $l > l_*$ ($0.75 < l < 1.29$), the exponent β is still close to 1/4, but the "breathing" and quadrupole modes mix. For σ far below σ_c the mode corresponding to the lowest frequency is the "breathing" mode 2. As we approach the critical value of σ the character of the lowest frequency mode changes and becomes quadrupole-like.

For l close to l^* ($l < 0.75$) the situation changes and the mode corresponding to the lowest frequency is the quadrupole mode 3. Now, it is its frequency ω_3 which tends to zero as the parameter σ approaches the critical value. For l not too close to l^* the exponent β is still close to 1/4. Completely new effects arise due to the existence of the previously discussed critical aspect ratio $l^* \simeq 0.41$. This is illustrated in Fig. 3. As one approaches l^*, the exponent β departs from 1/4 towards a greater value. This crossover is explained in Fig. 3. For l slightly below l^*, the frequency of the quadrupole mode ω_3 has a quadratic minimum close $\sigma = \sigma_c(l^*)$. Exactly at $l = l^*$, ω_3 goes thus to zero as $(\sigma_c - \sigma)^2$, i.e. $\beta = 2$.

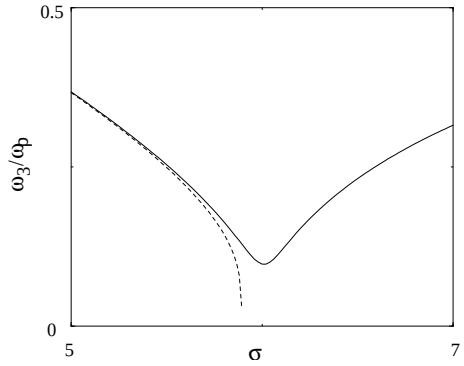

FIG. 3. Frequency ω_3 of the lowest mode 3 as a function of σ for l just below (solid line), and just above (dashed line) l^*. For $l < l^*$, the frequency ω_3 has a quadratic minimum close to $\sigma = \sigma_c(l^*)$.

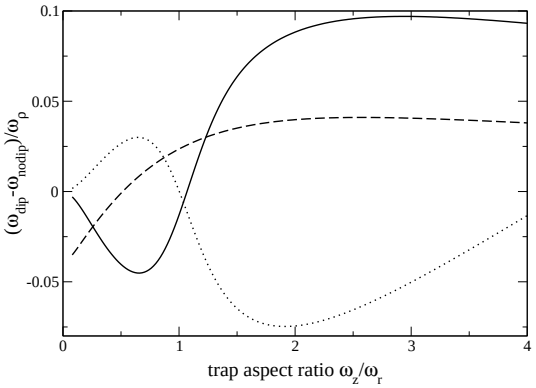

FIG. 4. Difference of excitation frequency for modes 1 (dashed line), 2 (dotted line) and 3 (solid line) between the cases of purely contact and mixed (contact and dipole-dipole) interactions as a function of the trap aspect ratio for 10000 atoms and $a = a_{\rm Na}$ (data from variational analysis).

Apart from the question of the nature of instability, in Ref. [30] we have considered also the situation in which the particles interact via both contact and dipole-dipole forces. In this case the strength for both types of forces is comparable and neither of them can be neglected. As a specific example we have considered the case of ^{52}Cr, which has drawn some experimental interest [33–36]. Chromium has a large magnetic dipole moment of $6\mu_B$ (Bohr magnetons), but its s -wave scattering length is still unknown. We have considered the trap frequencies of an experiment underway at the University of Stuttgart [37]: $\omega_z = 2\pi 40$ Hz and $\omega_\rho = 2\pi 485$ Hz. Our results suggest clearly that the effects of dipole-dipole interaction would be accessible to experimental detection if condensation of chromium or europium [38] is

achieved, and if their scattering lengths are not anomalously large. Dipole-dipole effects can be enhanced by reducing the s-wave scattering by employing Feshbach resonances [39,40]. It is thus very important to analyze possible parameters (scattering length, number of atoms, trap aspect ratio) that would maximize the predicted frequency shifts. A typical results of our analysis is shown in Fig. 4.

III. DIPOLAR FERMI GASES

The recent success in observing the quantum degeneracy in ultra-cold atomic Fermi gases [41–44] stimulates a search for gaseous Fermi systems where combined effects of Fermi statistics and interparticle interactions result in a non-trivial physical behavior. Due to the Pauli principle, to observe these effects in the case of a short-range Van der Waals interparticle interaction, a simultaneous trapping of at least two different fermionic species is needed, with a rather severe constraint on their relative concentrations. The situation is different for Fermi gases of dipolar particles. In the ultra-cold limit, the dipole-dipole scattering amplitude is energy independent for any orbital angular momenta $L > 0$. This observation follows from the results of [45,46]. It has also been recently discussed with important physical insight and consequences by You and coworkers [47,48]. This opens prospects to observe the effects of interparticle interaction in a *single-component* Fermi gas, where only scattering with odd orbital momenta (negligible in the case of Van der Waals interactions) is present. These prospects are especially interesting as in single-component fermionic gases the Pauli exclusion principle provides a strong suppression of inelastic collisional rates (see [46]). Hence, one can think of achieving significantly higher densities than in Bose gases.

A. Physical system

We consider a single component gas of fermionic dipolar particles in a harmonic trap, with dipole moments oriented in the same direction (z -axis). Then, due to the Pauli principle, the contribution of the short-range part of the interparticle interaction can be neglected, and, therefore, the dominant interaction between particles is the dipole-dipole one. As a result, the Hamiltonian of the system reads

$$H = \int d\mathbf{r}\hat{\psi}^\dagger(\mathbf{r})\left\{-\frac{\hbar^2}{2m}\Delta + V_{\rm trap}(\mathbf{r}) - \mu\right\}\hat{\psi}(\mathbf{r}) \qquad (3)$$

$$+\frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})V_{\rm d}(\mathbf{r}-\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r}'), \qquad (4)$$

where $V_{\rm d}(\mathbf{r}) = (d^2/r^3)(1-3\cos^2(\theta_{\mathbf{r}}))$ is the dipole-dipole interaction with $\theta_{\mathbf{r}}$ being the angle between the interparticle distance $\mathbf{r}$ and the z-axis, μ is the chemical potential, and $V_{\rm trap}(\mathbf{r}) = m[\omega_z^2 z^2 + \omega_\rho^2(x^2+y^2)]/2$ is the trapping potential; $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^\dagger(\mathbf{r})$ denote annihilation and creation operators for fermions with canonical anticommutation relations.

We assume that the gas is in the regime of quantum degeneracy, i.e. the temperature T is much smaller than the chemical potential μ (or the Fermi energy ε_F), $T \ll \mu = \varepsilon_F = (\hbar^2/2m)(6\pi^2 n_0)^{2/3}$, where n_0 is the maximal gas density (in the center of the trap), and the interparticle interaction is weak. The latter means that the mean dipole-dipole interaction energy per particle nd^2 is much less than the Fermi energy, $nd^2 \ll \varepsilon_F$.

B. Why are dipolar Fermi gases interesting?

Which features make this system attractive and challenging for both theorists and experimentalists? The system under consideration is unique in that the physics is dominated by the long-range dipole-dipole forces (the short-range part of the interparticle interaction, as already mentioned, can be neglected due to the fermionic nature of the particles). The dipole-dipole interaction is anisotropic and, as a result, it mixes scattering channels with different angular momenta l. Additionally, the interaction is partially repulsive (when two particles are side by side to each other) and partially attractive (when they are on top of each other). This means that, similarly as for dipolar Bose gases, the properties of the system depend on the trap geometry (see Ref. [49]). Indeed, in a pancake trap, most of the time the particles are side by side to each other, and the average interparticle interaction is repulsive. In the opposite case of a cigar shaped trap, the particles are mostly on top of each other and, as a result, the average interaction is attractive. Therefore, one can expect a pronounced dependence of the system properties on the aspect ratio $l = \sqrt{\omega_\rho/\omega_z}$ of the trap.

The fact that the dipole-dipole interparticle interaction is partially attractive opens the possibility for the BCS pairing at sufficiently low temperatures. For example, the matrix element $\langle L=1, M=0\,|V_{\rm d}|\,L=1, M=0\rangle$ of the dipole-dipole interaction between states with angular momentum $L=1$ and its projection to the z-axis $M=0$, is negative:

$$\langle L=1, M=0\,|V_{\rm d}|\,L=1, M=0\rangle = -\frac{4\pi}{5}d^2 < 0.$$

This signals about the possible BCS transition [50–52].

C. The BCS pairing

The BCS pairing transition corresponds to the formation of Cooper pairs, the correlated states of two fermions. The corresponding order parameter Δ that appears below the transition temperature T_c, describes the coherent motion of the Cooper pairs:

$$\Delta(\mathbf{r}_1, \mathbf{r}_2) \propto \langle \psi(\mathbf{r}_1)\psi(\mathbf{r}_2)\rangle \neq 0.$$

The order parameter vanishes for $T \to T_c$ and obeys the gap equation. For temperatures close to T_c this equation has the form (the so-called linearized gap equation):

$$\Delta(\mathbf{r}_1, \mathbf{r}_2) = -V_{\rm d}(\mathbf{r}_1 - \mathbf{r}_2) \int d\mathbf{r}_3 d\mathbf{r}_4 K(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4)\Delta(\mathbf{r}_3, \mathbf{r}_4), \tag{5}$$

where the kernel K is defined as

$$K(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4) = \sum_{\nu_1,\nu_2} \frac{\tanh(\xi_{\nu_1}/2T) + \tanh(\xi_{\nu_2}/2T)}{\xi_{\nu_1} + \xi_{\nu_1}} \phi_{\nu_1}(\mathbf{r}_1)\phi_{\nu_2}(\mathbf{r}_2)\phi^*_{\nu_1}(\mathbf{r}_3)\phi^*_{\nu_2}(\mathbf{r}_4)$$

with $\xi_\nu = \varepsilon_\nu - \mu$ and $\phi_\nu(\mathbf{r})$ being the solutions of the single-particle Schrödinger equation

$$\left\{-\frac{\hbar^2}{2m}\Delta + V_{\rm trap}(\mathbf{r})\right\}\phi_\nu(\mathbf{r}) = \varepsilon_\nu \phi_\nu(\mathbf{r}).$$

The critical temperature T_c can be defined as the maximum temperature for which the linearized gap equation (5) has a nontrivial solution. It should be mentioned that this equation can only be used for finding the critical temperature and the spatial dependence of the order parameter. To determine the absolute value of the order parameter, which is temperature dependent, one has to consider the terms nonlinear in Δ, omitted in Eq.(5).

D. Spatially homogeneous dipolar Fermi gases

In the spatially homogeneous case ($V_{\rm trap}(\mathbf{r})$ =0), the order parameter depends only on the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\Delta(\mathbf{r}_1, \mathbf{r}_2) = \Delta(\mathbf{r}_1 - \mathbf{r}_2)$, and the gap equation (5) with the leading non-linear term added, was discussed in Ref. [52], where it was found that the critical temperature of the BCS transition in the spatially homogeneous Fermi gas of dipolar particles equals (including the so-called Gor'kov and Melik-Barkhudarov corrections)

$$T_c = 1.44\varepsilon_F \exp(-\frac{\pi\varepsilon_F}{12nd^2}) = 1.44\varepsilon_F \exp(-\frac{\pi\hbar}{2\,|a_d|\,p_F}), \tag{6}$$

where n is the gas density, $\varepsilon_F = p_F^2/2m = (\hbar^2/2m)(6\pi^2 n)^{2/3}$ the Fermi energy, p_F the Fermi momentum, and $a_d = -2md^2/\pi^2\hbar^2$ the effective scattering length. The latter has been introduced in order to make the exponent in the expression for T_c, Eq.(6), look like the exponent in the expression for the critical temperature in the case of a two-component Fermi gas with the inter-component s-wave scattering length a_d. For the case of a single-component gas of fermionic ND_3 molecules with the dipole moment $d = 1.5$D, the transition temperature T_c is larger than 100 nK at densities $n > 5 \cdot 10^{12}$ cm^{-3}.

The properties of the superfluid dipolar Fermi gas are different from those of a two-component fermionic gas with s-wave pairing due to a short-range inter-component interaction. In the case of the s-wave paring, the order parameter is isotropic, whereas it is anisotropic in the superfluid dipolar Fermi gas. In the latter case, the value of the order parameter Δ at the Fermi surface $p = p_F$, is

$$\Delta(p_F, \theta) = 2.5T_c\sqrt{(T_c - T)/T_c} \cdot \sqrt{2}\sin(\frac{\pi}{2}\cos(\theta)),$$

where θ is the angle between the momentum $\mathbf{p}$ and the z-axis. As a result, the gap in the spectrum of single-particle excitations, which appears below the transition temperature T_c, is anisotropic. For example, excitations with momenta in the direction of the dipoles acquire the largest gap. In contrast to this, the eigenenergies of excitations with momenta perpendicular to the dipoles remain unchanged. The properties of collective excitations are also expected to be dependent on the direction of their momenta. Therefore, the response of the dipolar superfluid Fermi gas to small external perturbations will have a pronounced anisotropic character.

Another distinguishing feature of the superfluid dipolar Fermi gas is related to the temperature dependence of the specific heat. Well below the critical temperature the single-particle contribution to the specific heat is proportional to T^2, rather than being exponentially small as in the case of the s-wave pairing. This follows from the fact that the energy ε of single-particle excitations has a line of zeros on the Fermi surface: $\varepsilon(p_F) = 0$ for the angles at which $\Delta(p_F, \mathbf{n}) = 0$, i.e. for $\theta = \pi/2$ and an arbitrary azimuthal angle φ. As a consequence, the density of states in the vicinity of the Fermi energy is $\nu(\varepsilon) \sim \varepsilon$ for $\varepsilon \ll \Delta_0$. Therefore, at temperatures $T \ll \Delta_0 \sim T_c$, the temperature dependent part of the energy of the system is proportional to T^3, and the specific heat is thus proportional to T^2. This contribution is much larger than that of collective modes, which is $\propto T^3$ and is dominant in the case of the s-wave pairing.

E. BCS pairing in a harmonic trap

We now discuss how the results of the previous Section change when the harmonic trapping potential is switched on. It turns out (see Ref. [53] for details), that the presence of the trapping potential always results in the decrease of the critical temperature of the superfluid tran-

sition. In the case where the trap frequencies ω_z, ω_ρ are much smaller than the critical temperature T_c, Eq.(6), of the transition in the spatially homogeneous gas with density n equal to the maximum gas density n_0 in the trap, one obtains

$$\frac{T_c^{\rm trap}-T_c}{T_c} = -\frac{\omega}{T_c}\sqrt{\frac{7\zeta(3)}{48\pi^2}\left(1+\frac{\pi\varepsilon_F}{24nd^2}\right)} \left\{2\sqrt{1-\frac{3}{\pi^3}}l^{2/3}+\sqrt{1+\frac{6}{\pi^3}}l^{-4/3}\right\}, \quad (7)$$

where $T_c^{\rm trap}$ is the transition temperature in the trap, $\zeta(z)$ the Riemann zeta-function, and $\omega=(\omega_z\omega_\rho^2)^{1/3}$. One can see from this expression that for sufficiently small ω/T_c, the critical temperature in the trap $T_c^{\rm trap}$ is only slightly lower than T_c. As follows from Eq.(7), the optimal value of the trap aspect ratio l, that corresponds to maximal $T_c^{\rm trap}$ at a given T_c, is $l\simeq 1.38$.

Another interesting feature of the BCS transition in a trapped Fermi gas of dipolar particles is the existence of the critical value for ω_z. This comes from the fact that the paired states in the trapped dipolar Fermi gas have different quantum number n_z, and, therefore, their energies are always different, at least by the amount ω_z. When the difference becomes of the order of the order parameter $\Delta\sim T_c$ which measures the strength of the paring correlations, the pairing is obviously impossible. As a result, the superfluid transition in the dipolar Fermi gas is possible only in traps with $\omega_z<\omega_{zc}$, where the critical frequency ω_{zc} is found to be $\omega_{zc}=1.8T_c$. As can be seen from Eq.(7), confinement in the radial direction decreases the critical temperature as well. Therefore, in general one would expect the existence of the critical aspect ratio lc such that the pairing is possible only in a trap with $l>l_c$.

IV. DIPOLAR BOSE GASES IN OPTICAL LATTICES

In a very recent paper [54], we have investigated the ground-state properties of a polarized gas of bosonic particles (atoms or molecules) possessing a dipole moment and placed in an optical lattice. This system presents features which are novel both in the theory of quantum phase transitions, and in the context of degenerate quantum gases. Recently, Jaksch *et al.* [12] analyzed the superfluid-Mott insulator phase transition in the context of cold bosonic atoms with short-range interactions in an optical lattice. This analysis has been very recently confirmed experimentally [11]. The dipole-dipole interactions, which are anisotropic and have a long range, have not been studied, to the best of our knowledge, in the context of the Mott insulator to superfluid phase transition. In addition, we shall show that the long-range character of the dipole-dipole interaction allows for the existence of not only Mott-insulating and superfluid phases, but also several other phases in the system. Very importantly, the gas of cold dipolar bosons in an optical lattice is shown to be a system with easily tunable interactions which may permit to realize all the different phases experimentally. The availability of such a highly controllable system may be crucial in answering some unresolved questions and controversies in the theory of quantum phase transitions (e.g. the existence of a yet unobserved supersolid [55], or a Bose metal at zero temperature [56]).

A dilute gas of bosons in a periodic potential (e.g. in an optical lattice) can be described with the help of the Bose-Hubbard model [12]. For particles interacting via long-range forces the Bose-Hubbard Hamiltonian becomes:

$$H = J\sum_{<i,j>} b_i^\dagger b_j + \frac{1}{2}U_0\sum_i n_i(n_i-1) + \frac{1}{2}U_1\sum_{<i,j>} n_i n_j + \frac{1}{2}U_2\sum_{<<i,j>>} n_i n_j + \ldots, \quad (8)$$

where b_i is an operator annihilating a particle at a lattice site i in a state described by the Wannier function $w(\mathbf{r}-\mathbf{r}_i)$ of the lowest energy band, localized on this site. $\mathbf{r}_i$ is the position of the local minimum of the optical potential, and $n_i=b_i^\dagger b_i$ is the number operator for the site i. In Eq. (8) only the nearest neighbor tunneling is considered, which is described by the parameter J. The interparticle interactions are characterized by the parameters U_m, where $m=|j-i|$. In particular, U_0 determines the on-site interactions, U_1 the nearest-neighbor interactions, U_2 - the interaction between the next-nearest neighbors, etc. Consequently, the respective summations in Eq.(8) must be carried out over appropriate pairs of sites which are marked by <> for the nearest neighbors, <<>> for the next-nearest neighbors, etc. Note that in 2D and 3D, i is a multi-index enumerating sites in the corresponding lattice.

If only short-range interactions are present, seminal theoretical studies of the Bose-Hubbard model and related theory of arrays of Josephson junctions [57], show that two different quantum phases can occur for bosons in a lattice, either superfluid or Mott-insulator. In the case of finite-range interactions additional phases as supersolid and checker-board are expected [58].

We have found the ground state of the system by employing a variational approach based in the so-called Gutzwiller Ansatz (see [12] and references therein). We have shown that by modifying well-controllable parameters a whole variety of different quantum phases can be achieved, including superfluid, supersolid, Mott-insulator, checker-board and collapse phases. This possibility of manipulation of the corresponding quantum phases by modifying easily controllable external parameters (e.g. the on-site aspect ratio) makes the dipolar Bose

gas in an optical lattice particularly challenging for theory and experiments. In particular, possible applications for quantum information processing are worth mentioning in this context [16,17].

V. PHYSICAL REALIZATIONS OF ULTRACOLD DIPOLAR GASES

One of the possible physical realizations of a gas of dipolar particles is provided by electrically polarized gases of polar molecules. This molecules can have a large permanent electric dipole. The creation of cold clouds of polar molecules has been recently demonstrated in experiments with buffer-gas cooling [59], as well as in experiments based on deceleration and cooling of polar molecules by time-dependent electric fields [60]. The molecular dipole moments typically range from 0.1 D (1 D= 10^{-18}charge SGS $\times$ cm) to 1 D. For bosonic molecules that should be sufficient for achieving BEC at relatively low density of the gas. For example, the dipole moment of the fermionic deuterated ammonia molecule $^{15}ND_3$ is $d = 1.5$ D, which corresponds to an effective scattering length (see above, Eq.(6)) of $a_{\rm d} = -1450$ Å. This is even larger than the scattering length for the inter-component interaction in the widely discussed case of fermionic ^{6}Li. On the other hand, for the fermionic molecule $^{14}N^{16}O$ with dipole moment $d = 0.16$ D, the corresponding scattering length $a_{\rm d} = -24$ Å.

Another possibility is to use magnetic atomic dipoles [33–38]. Chromium atoms have a magnetic moment $\mu = 6\mu_B$, which is equivalent to an electric dipole moment of $d^* = 6\cdot 10^{-2}$ D, and an effective scattering length $a_{\rm d} = -5$ Å. This is too small and is unlikely to result in any interesting fermionic effects, such as BCS pairing. As we mentioned in section II, however, this could be enough to observe the influence of dipole-dipole interactions on the elementary excitations of Chromium BEC, provided the (still unknown) s-wave scattering length is not anomalously large, and it takes a value in the range of few tens of Å.

Permanent electric dipole moments can also be created by applying a high dc electric field to an atom. This possibility was discussed in Ref. [47], and here we only mention that in order to induce the dipole moment of the order of 0.1 D (the corresponding scattering length $a_d \sim -10 \div 100$ Å) one needs an electric field of the order of 10^6 V/cm. Nevertheless, the influence of dipole forces on elementary excitation spectrum of a dipolar BEC might be in this case also perhaps observable using not as high electric fields.

Finally, we mention the possibility of inducing a time averaged electric dipole moment of an atom by a stroboscopic laser coupling of the ground atomic state to a Rydberg state in a moderate dc electric field. This method has been proposed in the context of bosonic gases in Ref. [24]. We propose to place a BEC of alkali atoms into a moderate static electric field. The idea is to admix, with the help of a laser, to the atomic ground state the permanent dipole moment of a low-lying Rydberg state. Rydberg states of Hydrogen and alkali atoms exhibit a linear Stark effect [61]: in Hydrogen, for example, an electric field E_s splits the manifold of a Rydberg states of given principal quantum number n and magnetic quantum number m into $2(n - |m| - 1)$ Stark states. The outermost Stark states have (large) permanent dipole moments $d_R \sim n^2 e a_B$ (with a_B the Bohr radius), and there will be an associated dipole-dipole force between atoms. The spacing $\hbar\omega_s \simeq n e a_B E_s$ between adjacent Stark states should greatly exceed the mean-field dipole-dipole interaction (and the gas temperature) in order to avoid interaction-induced transitions from the lowest sublevel to other sublevels of the manifold.

This dipole-dipole interaction can be controlled with a laser [16]. This is achieved either by admixing the permanent dipole moment of the Stark states to the atomic ground state with an off-resonant cw laser, or by a stroboscopic excitation with a sequence of laser pulses. The pulses should be separated by the time T, have duration $2\Delta t \ll T$ and area multiple of 2π.

The field E_s and the laser should be chosen such that they do not couple the selected lowest sublevel to other Rydberg (sub)states. The stroboscopic excitation "dresses" the atomic internal states, so that each atom acquires a time averaged dipole moment of the order of $d_s = n^2 e a_B f$, oriented in the direction of E_s, where $f = \Delta t/T$. Even though the quantity f is assumed to be small, the induced dipole can be rather large for $n \gg 1$. Taking for example $\Delta t = 1$ns, $T = 10\mu$s, and $n = 20$, we obtain $d_s = 0.1$D. The resulting time dependent Hamiltonian can be replaced by its time average, leading to Eq.(1) with $d = d_s$. The characteristic time scale in Eq.(1) is provided by the inverse of the trap frequency ω^{-1}. Hence, in our case the dynamics of the system is described by Eq.(1) with $d = d_s$, if the condition $\Delta t, T \ll \omega^{-1}$ is satisfied.

It is important to note that, for the values of the scattering length of the order of $-10 \div 100$ Å, one nevertheless has a possibility to achieve the critical temperature of the BCS transition of the order of 100 nK at densities $n \sim 10^{16}$ cm^{-3}. In the case of a single-component atomic Fermi gas, such densities are not unrealistic because the inelastic processes, such as two-body collisions and three-body recombination, that usually limit the maximum value of the gas density, are strongly suppressed. This comes from the fact that these processes take place at short interatomic distances (of the order of tens of Angstroms) where the wave function of the relative motion of two identical fermions vanishes due to the Pauli principle.

VI. CONCLUSIONS

In this paper we have reviewed the properties of the bosonic and fermionic dipolar gases, and analyzed the perspectives that these systems can offer. We have shown, that the physics of these systems can differ qualitatively in a significant way with respect to the gases interacting via Van der Waals forces. In addition, new easy ways of control and manipulation are possible for dipolar gases. For all of that, we consider that the ultracold dipolar quantum gases constitute an interesting challenge for theory and experiments in AMO and condensed-matter physics.

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[1] 1987 Nobel Prize to J. Georg Bednorz and K. Alexander Müller, for their important break-through in the discovery of superconductivity in ceramic materials; 1996 Nobel Prize to D. M. Lee, D. D. Osheroff, and R. C. Richardson for their discovery of superfluidity in helium-3; related prize for studies of quantum Hall effect was given in 2000 to R. B. Laughlin, H. L. Störmer, and D. C. Tsui.

[2] In AMO physics W. Paul obtained the 1989 Nobel Prize for his studies on ion traps; S. Chu, C. Cohen-Tannoudji, and W. D. Phillips were awarded in 1997 for their work on laser cooling; in 2001, E. A. Cornell, W. Ketterle, and C. E. Wieman got the Nobel Prize for their contributions to the field of BEC.

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